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NOTE ON CONGRUENCE CLASSES OF $n-$GROUPS

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Abstract. In the paper the following proposition is proved. Let (Q, A) be an $n-$group, $|Q| \in N\backslash\{1\}$, and let $n \geq 3$. Further on, let Θ be an arbitrary congruence of the $n-$group (Q, A) and let C_t be an arbitrary class from the set Q/Θ. Then there is a $k \in N$ such that the pair $(C_t, \overset{k}{A})$ is a $(k(n-1)+1)-$subgroup of the $(k(n-1)+1)-$group $(Q, \overset{k}{A})$.

1. Preliminaries

1.1. Definition: *Let (Q, A) be an $n-$groupoid and $n \geq 2$. We say that (Q, A) is a Dörnte $n-$group [briefly: $n-$group] iff is $n-$semigroup and an $n-$quasigroup as well.*

1.2. Proposition [8]: *Let (Q, A) be an $n-$groupoid and $n \geq 2$. Then the following statements are equaivalent: (i) (Q, A) is an $n-$group; (ii) there are mappings $^{-1}$ and* $\mathbf{e}$*, respectively, of the sets Q^{n-1} and Q^{n-2} into the set Q such that the following laws hold in the algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$ [of the type $< n, n-1, n-2 >$]*

(a) $A(x_1^{n-2}, A(x_{n-1}^{2n-2}), x_{2n-1}) = A(x_1^{n-1}, A(x_n^{2n-1})),$

(b) $A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = x$ *and*

(c) $A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a) = \mathbf{e}(a_1^{n-2})$; *and*

(iii) there are mappings $^{-1}$ and $\mathbf{e}$*, respectively, of the sets Q^{n-1} and Q^{n-2} into the set Q such that the following laws hold in the algebra $(Q, \{\cdot, \varphi, b\})$ [of the type $< n, n-1, n-2 >$]*

$(\overline{a})$ $A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-1}),$

$(\overline{b})$ $A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x$ *and*

$(\overline{c})$ $A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = \mathbf{e}(a_1^{n-2}).$

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1.3. Remark: **e** *is an* $\{1,n\}-$*neutral operation of* $n-$*groupoid* (Q,A) *iff algebra* $(Q,\{A,\mathbf{e}\})$ *of the type* $< n, n-2 >$ *satisfies the laws* (b) *and* $(\overline{b})$ *from 1.2[4].*

1.4. Proposition (Hosszú–Gluskin Theorem) [2, 3]: *For every* $n-$*group* (Q,A)*,* $n \geq 3$*, there is an algebra* $(Q,\{\cdot,\varphi,b\})$ *such that the following statements hold:* 1° $(Q,\cdot)$ *is a group;* 2° $\varphi \in Aut(Q,\cdot)$; 3° $\varphi(b) = b$; *for every* $x \in Q$*,* $\varphi^{n-1}(x) \cdot b = b \cdot x$; *and* 5° *for every* $x_1^n \in Q$*,* $A(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \dots \cdot \varphi^{n-1}(x_n) \cdot b$.

1.5. Definition [5]: *We say that an algebra* $(Q,\{\cdot,\varphi,b\})$ *is a Hosszú–Gluskin algebra of order* $n(n \geq 3)$ *[briefly:* $nHG-$*algebra] iff* $1^\circ - 4^\circ$ *from 1.4 hold. In adittion, we say that an* $nHG-$*algebra* $(Q,\{\cdot,\varphi,b\})$ *is associated to the* $n-$*group* (Q,A) *iff* 5° *from 1.4 holds.*

2. Auxiliarly proposition

2.1. Proposition [5]: *Let* (Q,A) *be an* $n-$*group,* **e** *its* $\{1,n\}-$*neutral operatopn [:1.3] and* $n \geq 3$*. Further on, let* c_1^{n-2} *be an arbitrary sequence over* Q *and let for every* $x, y \in Q$

(1) $B_{(c_1^{n-2})}(x) \stackrel{def}{=} A(x, c_1^{n-2}, y)$,

(2) $\varphi_{(c_1^{n-2})}(x,y) \stackrel{def}{=} A(\mathbf{e}(c_1^{n-2}), x, c_1^{n-2})$ *and*

(3) $b_{(c_1^{n-2})} \stackrel{def}{=} A(\overset{n}{\overline{\mathbf{e}(c_1^{n-2})}}|)$.

Then, the following statements hold:

(4) $(Q,\{B_{(c_1^{n-2})}, \varphi_{c_1^{n-2})}, b_{(c_1^{n-2})}\})$ *is an* $nHG-$*algebra associated to the* n*-group* (Q,A); *and*

(5) $\mathcal{C}_A \stackrel{def}{=} \{(Q,\{B_{(c_1^{n-2})}, \varphi_{(c_1^{n-2})}, b_{(c_1^{n-2})}\}) | c_1^{n-2} \in Q\}$ *is the set of all* $nHG-$*algebras associated to the* $n-$*group* (Q,A)*.*

2.2. Proposition [10]: *Let* (Q,A) *be an* $n-$*group and let* $n \geq 3$*. Further on, let* Θ *be an arbitrary congruence of the* $n-$*group* (Q,A)*. Then, for every* $C_t \in Q/\Theta$ *there is an* $nHG-$*algebra* $(Q,\{\cdot,\varphi,b\})$ *associated to the* $n-$*group* (Q,A) *such that the following statements hold:*

(i) $(C_t,\cdot) \lhd (Q,\cdot)$;

(ii) (C_t,φ) *is a 1–quasigroup; and*

(iii) (C_t,A) *is an* $n-$*subgroup of the* $n-$*group* (Q,A) *iff* $b \in C_t$.

2.3. Remark: *Let* **e** *be a* $\{1,n\}-$*neutral operation of the* $n-$*group* (Q,A) *[:1.3]. Then: a) for* $n = 3$*,* $\mathbf{e} \in Q!$; *and b) for* $n > 3$*,* $(Q,\mathbf{e})$ *is an*

$(n-2)$-quasigroup [5]. $nHG-$algebra in Prop. 2.2. is defined with $\mathbf{e}(c_1^{n-2}) = t$ and with (1) – (3) from Prop. 2.1.

2.4. Proposition: *Let (Q,A) be an $n-$semigroup, $n \geq 2$ and let $(i,j) \in N^2$. Let also $\overset{1}{A} \overset{def}{=} A$ and for every $m \in N$ and for every $x_1^{(m+1)(n-1)+1} \in Q$*

$$\overset{m+1}{A}(x_1^{(m+1)(n-1)+1}) \overset{def}{=} A(\overset{m}{A}(x_1^{m(n-1)+1}), x_{m(n-1)+2}^{(m+1)(n-1)+1}).$$

Then, for every $x_1^{(m+1)(n-1)+1} \in Q$ and for every $t \in \{1, \dots, i(n-1)+1\}$, the following equality holds

$$\overset{i}{A}(x_1^{t-1}, \overset{j}{A}(x_t^{t+j(n-1)}), x_{t+j(n-1)+1}^{(i+j)(n-1)+1}) = \overset{i+j}{A}(x_1^{(i+j)(n-1)+1}).$$

By 2.4 an by 1.1, we conclude that the following proposition holds:

2.5. Proposition: *Let (Q,A) be an $n-$group, $n \geq 2$ and $i \in N$. Then $(Q, \overset{i}{A})$ is an $(i(n-1)+1)-$group.*

2.6. Proposition [11]: *Let (Q,A) be an $n-$group and $n \geq 2$. Then for every $k \in N \backslash \{1\}$ the following equality holds $Con(Q,A) = Con(Q, \overset{k}{A})$.*

3. Result

3.1. Theorem: *Let (Q,A) be an $n-$group, $|Q| \in N\backslash\{1\}$ and let $n \geq 3$. Further on, let Θ be an arbitrary congruence of the $n-$group (Q,A) $[\Theta \in Con(Q,A)]$ and let C_t $[t \in Q]$ be an arbitrary class from the set Q/Θ. Then there is a $k \in N$ such that the pair $(C_t, \overset{k}{A})$ is a $(k(n-1)+1)-$ subgroup of the $(k(n-1)+1)-$group $(Q, \overset{k}{A})$.*

Proof. Firstly, we prove that under the assumptions the following statements hold:

$\overline{1}$ If $(Q, \{\cdot, \varphi, b\})$ is an $nHG-$algebra [:1.5], then for every $k \in N$ $(Q, \{\cdot, \varphi, b^k\})$ is a $(k(n-1)+1)HG-$algebra; and

$\overline{2}$ If $(Q, \{\cdot, \varphi, b\})$ is an $nHG-$algebra associated to the $n-$group (Q,A), then for every $k \in N$ $(Q, \{\cdot, \varphi, b^k\})$ is a $(k(n-1)+1)HG-$algebra associated to the $(k(n-1)+1)-$group $(Q, \overset{k}{A})$[1)].

The sketch of the proof of $\overline{1}$:

a) $\varphi(b^1) = b,\ \varphi(b^t) = b^t,$
$\varphi(b^{t+1}) = \varphi(b^t) \cdot \varphi(b) = b^t \cdot b = b^{t+1};$

[1)]See 2.5.

b) $\varphi^{t(n-1)}(x)\cdot b^t = b^t\cdot x$

$$\begin{aligned}\varphi^{(t+1)(n-1)}(x)\cdot b^{t+1} &= \varphi^{t(n-1)}(\varphi^{n-1}(x))\cdot b^t\cdot b\\ &= b^t\cdot\varphi^{n-1}(x)\cdot b\\ &= b^t\cdot b\cdot x\\ &= b^{t+1}\cdot x\ \ [:1.5].\end{aligned}$$

The sketch of the proof of $\overline{2}$:

$\overline{a}$) $\overset{1}{A}\overset{2\cdot 4}{=}A,\ A(x_1^n)\overset{1\cdot 5}{=}x_1\cdot\varphi(x_2)\cdot\dots\cdot\varphi^{n-1}(x_n)\cdot b;$

$\overline{b}$) $\overset{t}{A}(x_1^{t(n-1)+1}) = x_1\cdot\varphi(x_2)\cdot\dots\cdot\varphi^{t(n-1)}(x_{t(n-1)+1})\cdot b^t;$

$\overline{c}$) $\overset{t+1}{A}(x_1^{(t+1)(n-1)+1})\overset{2\cdot 4}{=}A(x_1^{n-1},\overset{t}{A}(x_n^{(t+1)(n-1)+1})) =$

$x_1\cdot\varphi(x_2)\cdot\dots\cdot\varphi^{n-2}(x_{n-1})\cdot\varphi^{n-1}\overset{t}{A}(x_n^{(t+1)(n-1)+1})\cdot b\overset{\overline{b})}{=}$

$x_1\cdot\varphi(x_2)\cdot\dots\cdot\varphi^{n-2}(x_{n-1})\cdot\varphi^{n-1}(x_n)\cdot\dots\cdot\varphi^{(t+1)(n-1)}(x_{(t+1)(n-1)+1})\cdot b^t\cdot b =$

$x_1\cdot\varphi(x_2)\cdot\dots\cdot\varphi^{n-2}(x_{n-1})\cdot\varphi^{n-1}(x_n)\cdot\dots\cdot\varphi^{(t+1)(n-1)}(x_{(t+1)(n-1)+1})\cdot b^{t+1}.$

Finally, by Proposition 2.6, by $\overline{1}$, $\overline{2}$, by $|Q|\in N\backslash\{1\}$, and by Proposition 2.2, we conclude that the Theorem holds. □

4. Remark

If $n\geq 3$,, then: 1) there exist $n-$group (Q,A) and its congruence Θ such that **for every** $C_a\in Q/\Theta$ the pair (C_a,A) **is not** an $n-$group [6,9]; 2) there exist an $n-$group (Q,A) and its congruence Θ such that **for every** $C_a\in Q/\Theta$ the pair (C_a,A) **is** an $n-$group [6,7,9]; and 3) there exist $n-$group (Q,A) and its congruence Θ such that **exactly one** $C_a\in Q/\Theta$ the pair (C_a,A) **is** an $n-$group [6].

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